LS- 95
W. Chou
August, 1987

AN ANALYTICAL ESTIMATION OF THE RMS TUNE SHIFTS DUE TO MAGNET IMPERFECTIONS

The tolerances of a storage ring to its magnet imperfections largely depend upon the magnitude of the resulting tune shifts. A large tune shift would imply the potential danger of resonances and, hence, poor tolerances. When the errors are in field gradients or in random sextupole displacement, the calculations of the RMS tune shifts are straight forward. On the contrary, when we consider the tune shifts due to errors in quadrupole placement or in dipole fields, the calculations are often confusing and mishandled. This note provides a concise formula for the RMS tune shifts due to either quadrupole displacement or dipole field errors. For simplicity, we will limit our discussions to the effects of quadrupole displacement. However, all our results should be applicable to the case where errors are in dipole fields.

The quadrupole displacement produces closed orbit (c.o.) distortions everywhere in a storage ring. In particular, the horizontal c.o. distortions at the sextupoles give rise to a tune shift. The calculations of these effects are well-known. For instance, the horizontal c.o. distortion at the j^{th} sextupole is

$$\Delta x_j = \frac{\sqrt{\beta_{xj}}}{2\sin\pi\nu_x} \sum_i \sqrt{\beta_{xi}}\, K_i \ell_i \cdot \cos(\mu_{xj} - \mu_{xi} - \pi\nu_x) \cdot \Delta_i \tag{1}$$

in which Δ_i is the displacement of the i^{th} quadrupole and $K_i = (B'/B\rho)_i$. The vertical c.o. distortion has a similar form. The horizontal and vertical tune shifts resulting from the sextupole displacement are

$$\Delta\nu_x = \frac{1}{4\pi} \sum_j \beta_{xj} S_j \Delta x_j, \qquad (2)$$

$$\Delta\nu_y = \frac{-1}{4\pi} \sum_j \beta_{yi} S_j \Delta x_j, \qquad (3)$$

respectively, with $S_j = (B''\ell/B\rho)_j$. These equations can be rewritten in the compact forms

$$\Delta x_j = \sum_i A_{ji} \Delta_i, \qquad (4)$$

$$\Delta\nu_x = \sum_j b_{xj} \Delta x_j, \qquad (5)$$

$$\Delta\nu_y = \sum_j b_{yj} \Delta x_j, \qquad (6)$$

where the coefficients

$$A_{ji} = \frac{\sqrt{\beta_{xj}}}{2 \sin\pi\nu_x} \sqrt{\beta_{xi}}\, K_i\, \ell_i \cdot \cos(\mu_{xj} - \mu_{xi} - \pi\nu_x), \qquad (7)$$

$$b_{xj} = \frac{1}{4\pi} \beta_{xj} S_j, \qquad (8)$$

$$b_{yj} = \frac{-1}{4\pi} \beta_{yj} S_j. \qquad (9)$$

When Δ_i's are totally random, one can define a (average) magnification factor, A_c, of the c.o distortions at sextupoles.

$$A_c = \left(\frac{1}{n_s} \sum_j \sum_i A_{ji}^{\;2}\right)^{1/2}, \qquad (10)$$

in which n_s is the total number of sextupoles. Similarly, in case Δx_j's are totally random (Warning! This is not the case if Δx_j's are c.o. distortions), the magnification factors of the tune shifts will be

$$A_{sx} = (\sum_j b_{xj}{}^2)^{1/2}, \tag{11}$$

and

$$A_{sy} = (\sum_j b_{yj}{}^2)^{1/2}. \tag{12}$$

Now, we consider the tune shifts due to quadrupole displacement. From Eqs. (4), (5) and (6), we get

$$\Delta\nu_x = \sum_j b_{xj} \sum_i A_{ji}\ \Delta_i$$

$$= \sum_i (\sum_j \ b_{xj}\ A_{ji})\Delta_i$$

$$= \sum_i c_{xi}\ \Delta_i, \tag{13}$$

and, similarly,

$$\Delta\nu_y = \sum_i c_{yi}\ \Delta_i, \tag{14}$$

in which

$$c_{xi} = \sum_j b_{xj}\ A_{ji}, \tag{15}$$

$$c_{yi} = \sum_j b_{yj} A_{ji}. \tag{16}$$

The parameter c_{xi} (c_{yi}) has a nice simple physical interpretation: it is the horizontal (vertical) tune shift resulting from a unit displacement of the i^{th} quadrupole. One can easily do statistics in view of Eqs. (13) and (14) to get the RMS tune shifts which are due to the random quadrupole displacement with an RMS value $\langle\Delta\rangle$.

$$\langle\Delta\nu_x\rangle = (\sum_i c_{xi}{}^2)^{1/2} \cdot \langle\Delta\rangle, \tag{17}$$

$$\langle\Delta\nu_y\rangle = (\sum_i c_{yi}{}^2)^{1/2} \cdot \langle\Delta\rangle. \tag{18}$$

One may then define the magnification factors

$$A_{qx} = (\sum_i c_{xi}{}^2)^{1/2}, \tag{19}$$

$$A_{qy} = (\sum_i c_{yi}{}^2)^{1/2}. \tag{20}$$

The values of the three different types of magnification factors discussed above have been calculated for the storage ring of the 7-GeV APS and are listed in Table 1. Note that

$$3_q << A_c \cdot A_s, \tag{21}$$

where A_q (A_s) stands for either A_{qx} (A_{sx}) or A_{qy} (A_{sy}).

Table 1

Magnification factors of the APS ring

	Horizontal, x	Vertical, y
A_c, for c.o distortion at sext. due to random quad. displacement	51	(not in use in this note)
A_s, tune shift due to random sext. displacement	40	66
A_q, tune shift due to random quad. displacement	230	120

Discussions:

1. A quite common mistake in calculating A_q is to take it simply as the product of A_c and A_s. This is wrong and often leads to a large overestimation. This can be seen in the following argument. From Eq. (5), one has

$$\Delta\nu_x^2 = \sum_j \sum_{j'} b_{xj}\, b_{xj'}\, \Delta x_j\, \Delta x_{j'}. \tag{22}$$

In order to define A_{sx}, Δx_j and $\Delta x_{j'}$ have to be statistically independent to each other (i.e., uncorrelated). If this is the case, one can then conclude that

$$\Delta\nu_x^2 = \sum_j b_{xj}^2 \cdot \Delta x_j^2, \tag{23}$$

which leads to the definition in Eq. (11) for A_{sx}. When, on the other hand, Δx_j and $\Delta x_{j'}$ are the c.o. distortions at the j^{th} and the j'^{th} sextupoles, they are not independent to each other. Rather, each one is derived from Eq. (4). Therefore, in order to do statistics, one has to go back to Eq. (4), replacing Δx_j and $\Delta x_{j'}$ in Eq. (22) by their expressions Eq. (4) in terms of Δ_i. One can then invoke the random nature of Δ_i to make statistical prediction for the RMS value of $\Delta\nu$. This is precisely what we did in the derivations of the formula in Eqs. (17) and (18) above.

2. There are some tricks in doing the double summation $\sum_j$ (in Eqs. (15) and (16)) and $\sum_i$ (in Eqs. (17) and (18)). To compute $\sum_j$, one has to sum over the whole ring, not just a period. In other words, $\sum_j$(over the ring) is not equal to $\sum_j$(over a period) multiplied by the number of periods. This is because that for a fixed i, the cosine function in Eq. (7) is not periodic in j. To compute $\sum_i$, on the other hand, one only needs to sum over a period and, then, to multiply the sum by the square root of the number of periods of the ring to get A_q. This is obvious in view of the physical meaning of c_{xi} (c_{yi}): the tune shift caused by (a unit displacement of) the i^{th} quadrupole in one period should be the same as that by the i^{th} quadrupole in another period.

3. Several programs run Monte Carlo simulations to get the RMS tune shifts for a given RMS value of quadrupole displacement. Unfortunately, sometimes different programs give quite different results. For example, RACETRACK, MAD, PETROS and PATRIS have been employed to determine the tolerances of the APS ring. For a 10^{-4} RMS quadrupole placement error, the results are listed in Table 2. The RMS tune shifts obtained by the first three programs above are fairly close to each other, whereas PATRIS gives the results which are bigger than those by an order of magnitude. Meanwhile, from the values of A_q listed in Table 1, our theoretical

predictions for the RMS tune shifts are also shown in Table 2, which asserts that the results from PATRIS are unlikely. On the other hand, these programs give a finite average tune shift $\Delta\bar{\nu}$ in addition to an RMS $<\Delta\nu>$ (the values of $\Delta\bar{\nu}$ are quite different from one program to the other), while Eqs. (13) and (14) (which are based on the first-order perturbation theory) would predict a zero value for $\Delta\bar{\nu}$ when Δ_i's are random. This discrepancy remains to be resolved.

4. As we have pointed out, all our results above are applicable to the case where the tune shifts are due to dipole field errors instead of quadrupole displacement. In this case, we replace $k_i \ell_i \Delta_i$ in (1) by $(\Delta B\ell/B\ell)_i$, the field error of the i^{th} dipole.

Table 2

Tune shifts of the APS ring for 10^{-4} RMS quadrupole displacement

	No. runs	Horizontal	Vertical	Horizontal	Vertical
Formula (13) & (14)	(∞)	0	0	0.023	0.012
RACETRACK	10	-0.032	0.016	0.043	0.018
MAD	4	-0.058	0.009	0.019	0.015
PETROS	10	-0.007	0.017	0.018	0.013
PATRIS	21	0.214	0.225	0.278	0.203

Notes: (1) Data of RACETRACK and MAD are obtained by Kramer, that of PETROS by Jin, and that of PATRIS by Chou.

(2) $\Delta\bar{\nu} = \sum_{i=1}^{n} \Delta\nu_i/n$

$$\langle\Delta\nu\rangle = \left(\sum_{i=1}^{n} (\Delta\nu_i - \Delta\bar{\nu})^2/n\right)^{1/2}$$